NASA Contractor Report 187583
ICASE Report No. 91-46

ICASE

A UNIQUENESS RESULT CONCERNING THE IDENTIFICATION OF A COLLECTION OF CRACKS FROM FINITELY MANY ELECTROSTATIC BOUNDARY MEASUREMENTS

Kurt Bryan
Michael Vogelius

Contract No. NAS1-18605
June 1991

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665-5225

Operated by the Universities Space Research Association

NASA
National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

A UNIQUENESS RESULT CONCERNING THE IDENTIFICATION OF A COLLECTION OF CRACKS FROM FINITELY MANY ELECTROSTATIC BOUNDARY MEASUREMENTS

Kurt Bryan[1]
Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA 23665

and

Michael Vogelius[2]
Department of Mathematics
Rutgers University
New Brunswick, N.J. 08903

A-1

ABSTRACT

We consider the problem of locating and identifying a collection of finitely many cracks inside a planar domain from measurements of the electrostatic boundary potentials induced by specified current fluxes. It is shown that a collection of n or fewer cracks can be uniquely identified by measuring the boundary potentials induced by $n + 1$ specified current fluxes, consisting entirely of electrode pairs.

[1]Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665-5225.
[2]Research is partially supported by National Science Foundation Grant DMS-89-02532 and AFOSR Contract 89-NM-605.

1 Introduction

In a recent paper, [1], A. Friedman and M. Vogelius proved that the presence of a single crack, its shape and location inside a planar domain may be determined from measurements of the steady state boundary voltage potentials corresponding to two specific boundary current fluxes. In the present paper we extend this result to any finite number of cracks: we show that voltage measurements corresponding to $n+1$ specific fluxes suffice to determine the location and shape of a collection of n (or fewer) cracks. In contrast to [1] the fluxes we use here all consist of electrode pairs – exactly the type of fluxes which are used for the computational algorithm developed in [2].

Let Ω be a simply connected domain in $\mathbb{R}^2$ with a smooth boundary. In order to describe our result in detail we need to define the notion of a collection of cracks. By a C^2-curve, σ, we understand a twice continuously differentiable map: $[0,1] \to \Omega$ with non-vanishing derivative. *A collection of cracks consists of a finite number of mutually disjoint, non-self-intersecting C^2-curves σ_k , $k = 1, \ldots, n$.* We use capital Greek letters to denote collections of cracks, e.g. $\Sigma = \{\sigma_k\}_{k=1}^n$; note that n may possibly be zero, so that Σ is empty. We shall also use the notation σ_k and Σ for the image of each of the individual curves and the union of all the images, respectively (*i.e.*, $\Sigma = \cup_{k=1}^n \sigma_k$). Let $\gamma : \overline{\Omega} \to \mathbb{R}$ be a positive function (the known reference conductivity). Throughout this paper we assume that

$$\gamma \text{ is real-analytic on } \overline{\Omega}.$$

In the following, when a function is referred to as being analytic, this shall

always mean real-analytic. Quite frequently in the literature the term crack is used synonymously with an electrically insulating crack: if ϕ represents the boundary voltage, then the steady state voltage potential satisfies

$$\begin{aligned}\nabla \cdot (\gamma \nabla v) &= 0 \text{ in } \Omega \setminus \Sigma, \\ \gamma \frac{\partial v}{\partial \nu} &= 0 \text{ on } \Sigma, \\ v &= \phi \text{ on } \partial\Omega.\end{aligned}$$

In this framework the inverse problem is to determine Σ from knowledge of several pairs $(\phi, \gamma \frac{\partial v}{\partial \nu}|_{\partial\Omega})$ We shall, instead of working with the potential v, opt to work with its "γ-harmonic" conjugate, u. This function is related to v by

$$(\nabla u)^{\perp} = \gamma \nabla v, \tag{1.1}$$

where $\perp$ indicates counter-clockwise rotation by $\pi/2$. Let T be a fixed point on $\partial\Omega$, in a neighborhood of which ϕ is smooth. Let τ_k be a smooth curve in $\Omega \setminus \Sigma$ connecting T to an interior point of the crack σ_k, and let s denote the unit tangent direction along τ_k, pointing from T towards σ_k. Define constants

$$c^{(k)} = \int_{\tau_k} \gamma \frac{\partial v}{\partial \nu} ds \; + \; u(T),$$

where ν denotes the normal field $\nu = s^{\perp}$. The "γ-harmonic" conjugate, u, solves

$$\begin{aligned}\nabla \cdot (\gamma^{-1} \nabla u) &= 0 \text{ in } \Omega \setminus \Sigma, \\ u &= c^{(k)} \text{ on } \sigma_k \;\; k = 1, \ldots, n \\ \gamma^{-1} \frac{\partial u}{\partial \nu} &= \frac{\partial \phi}{\partial s} = \psi \text{ on } \partial\Omega,\end{aligned} \tag{1.2}$$

where s denotes the counter-clockwise tangent direction on $\partial\Omega$.

From (1.1) it follows immediately that knowledge of $u|_{\partial\Omega}$ leads to knowledge of $-\gamma\frac{\partial v}{\partial\nu}|_{\partial\Omega} = \frac{\partial}{\partial s}(u|_{\partial\Omega})$, and vice versa. Therefore knowledge of pairs $(u|_{\partial\Omega}, \psi)$ is equivalent to knowledge of corresponding pairs $(\phi, \gamma\frac{\partial u}{\partial\nu}|_{\partial\Omega})$, where ϕ and ψ are related by $\psi = \frac{\partial\phi}{\partial s}$. Physically (1.2) corresponds to a collection of perfectly conducting cracks. One way to solve (1.2) is to minimize the energy

$$\frac{1}{2}\int_\Omega \gamma^{-1}|\nabla w|^2\, dx - \int_{\partial\Omega} \psi w\, ds$$

in the space $H^1(\Omega) \cap \{w = \text{ const on each } \sigma_k \in \Sigma\}$ (such minimization gives, modulo a single undetermined constant, exactly the values on σ_k, $k = 1, \ldots, n$ described above). This method works provided $\psi \in H^{-1/2}(\partial\Omega)$. The fluxes we shall apply here, however, correspond to single pairs of electrodes, *i.e.*, we shall take ψ of the form $\psi = \delta_{P_0} - \delta_{P_1}$, where P_0 and P_1 are two distinct points on $\partial\Omega$. Such ψ are not in $H^{-1/2}(\partial\Omega)$ – the solution, u, is therefore not in $H^1(\Omega)$ and it is not obtained as a minimizer of energy. Rather, u is a weak solution to (1.2); it is smooth everywhere except at P_0 and P_1 and at the endpoints of the cracks. At P_0 and P_1 the function u has singularities of the form $-\gamma(P_0)/\pi \ \log r$, $r = |x - P_0|$, and $\gamma(P_1)/\pi \ \log r$, $r = |x - P_1|$, respectively, at the endpoints of the cracks u has in general $r^{1/2}$-type singularities, cf. [1].

For our uniqueness result it is not necessary that we have solutions which attain exactly the constant values on the cracks described above – any set of constants will do. To construct the specific boundary currents, let $P_0, \ldots, P_M$ be $M + 1$ different (fixed) points on $\partial\Omega$; we assume that these points are

labeled in order of counter-clockwise appearance, starting from P_0. In our first uniqueness result we utilize solutions to the boundary value problems

$$\begin{aligned} \nabla \cdot (\gamma^{-1} \nabla u_j) &= 0 \text{ in } \Omega \setminus \Sigma, \\ u_j &= \text{constant on each } \sigma_k \in \Sigma \\ \gamma^{-1} \frac{\partial u_j}{\partial \nu} &= \delta_{P_0} - \delta_{P_j} \text{ on } \partial\Omega. \end{aligned} \tag{1.3}$$

Theorem 1.1 *Let* $\Sigma = \{\sigma_k\}_{k=1}^n$ *and* $\tilde{\Sigma} = \{\tilde{\sigma}_k\}_{k=1}^m$ *denote two collections of cracks contained in the domain* Ω, *with* $\max(m,n) + 1 \leq M$. *Let* u_j, $j = 1, \ldots, M$ *denote solutions to (1.3) and let* $\tilde{u}_j$, $j = 1, \ldots, M$ *denote solutions to (1.3) with* Σ *replaced by* $\tilde{\Sigma}$. *Then* $u_j = \tilde{u}_j$ *on* $\partial\Omega \setminus \cup_{i=0}^M \{P_i\}$ *for* $j = 1, \ldots, M$ *implies that* $\Sigma = \tilde{\Sigma}$.

Instead of prescribing fixed fluxes $\gamma^{-1} \frac{\partial u_j}{\partial \nu}|_{\partial\Omega} = \psi_j$ and measuring $u_j|_{\partial\Omega}$ we can equally well prescribe fixed boundary voltages $w_j|_{\partial\Omega} = \phi_j$ and measure $\gamma^{-1} \frac{\partial w_j}{\partial \nu}|_{\partial\Omega}$. For that purpose we utilize solutions to the following boundary value problems

$$\begin{aligned} \nabla \cdot (\gamma^{-1} \nabla w_j) &= 0 \text{ in } \Omega \setminus \Sigma, \\ w_j &= \text{constant on each } \sigma_k \in \Sigma, \\ w_j &= 1_{P_{j-1},P_j} \text{ on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $1_{P_{j-1},P_j}$ denotes the characteristic function of the counter-clockwise curve from P_{j-1} to P_j. The function w_j is a weak solution to (1.4) – it is not in $H^1(\Omega)$, and therefore not obtained as a minimizer of energy. w_j has a singularity of the form $-\theta/\pi$ at P_{j-1}, $\theta = \arg(x - P_{j-1})$ and has a singularity of the form θ/π at P_j, $\theta = \arg(x - P_j)$. At the endpoints of the cracks w_j has in general $r^{1/2}$-type singularities.

Theorem 1.2 *Let $\Sigma = \{\sigma_k\}_{k=1}^n$ and $\tilde{\Sigma} = \{\tilde{\sigma}_k\}_{k=1}^m$ denote two collections of cracks contained in the domain Ω, with $\max(m,n)+1 \leq M$. Let w_j, $j = 1,\ldots,M$ denote solutions to (1.4) and let $\tilde{w}_j$, $j=1,\ldots,M$ denote solutions to (1.4) with Σ replaced by $\tilde{\Sigma}$. Then $\gamma^{-1}\frac{\partial w_j}{\partial\nu} = \gamma^{-1}\frac{\partial \tilde{w}_j}{\partial\nu}$ on $\partial\Omega \setminus \cup_{i=0}^M\{P_i\}$ for $j=1,\ldots,M$ implies that $\Sigma = \tilde{\Sigma}$.*

REMARK: As was the case with the first theorem, this second theorem also has an alternative formulation in terms of cracks that are insulating. In that case one would prescribe boundary fluxes $-\partial(1_{P_{j-1},P_j})/\partial s = \delta_{P_j} - \delta_{P_{j-1}}$ and measure the corresponding boundary voltages.

2 Preliminaries

The proof of our main results consists in a very detailed analysis of the structure of the level curves of solutions to the equation $\nabla\cdot(\gamma^{-1}\nabla u) = 0$. For that purpose we shall need two auxiliary lemmas.

Lemma 2.1 *Let u satisfy $\nabla\cdot(\gamma^{-1}\nabla u) = 0$ in $\Omega\backslash\Sigma$ with $\gamma^{-1}\partial u/\partial\nu = \sum_j \beta_j\delta_{P_j}$ on $\partial\Omega$, and u constant on each σ_k. Let ρ be a non-empty analytic curve in Ω with $\rho\cap\Sigma = \emptyset$ along which u is constant. Then there exists an analytic curve ρ' with $\rho\subset\rho'$ such that*

(2.1a) *u is constant on ρ'*

(2.1b) *ρ' has one endpoint on $\partial\Omega$ or σ_k for some k*

(2.1c) *ρ' has the other endpoint on $\partial\Omega$ or σ_l for some l with $l\neq k$.*

The proof of this lemma is identical to the proof of lemma 2.3 in [1]. We shall not repeat the the proof here. The second lemma we need concerns the existence of intersecting level curves. Some of the details of the proof of this result are not unlike those found in the proof of lemma 2.3 in [1], but for the convenience of the reader we give a complete proof here.

Lemma 2.2 *Let u satisfy $\nabla\cdot(\gamma^{-1}\nabla u) = 0$ in $\Omega\backslash\Sigma$ with $\gamma^{-1}\partial u/\partial\nu = \sum_j \beta_j\delta_{P_j}$ on $\partial\Omega$, and u constant on each σ_k. Let ρ be a non-empty analytic curve in Ω with $\rho\cap\Sigma = \emptyset$ along which u is constant, and assume that x^* is an interior point of ρ where $\nabla u(x^*) = 0$. Then there exists an analytic curve ρ' which has x^* as an interior point such that*

(2.2a) $\rho'\cap\rho = \{x^*\}$

(2.2b) *u is constant on ρ'.*

Proof: Let $(r,\theta) \in [0,\epsilon] \times [0,2\pi]$ denote polar coordinates at x^*. Since $\nabla u(x^*) = 0$ we know that $\frac{\partial}{\partial r}u(0,\theta) \equiv 0$, and by expanding in a Taylor series in r we get

$$u(x) = u(x^*) + r^N(a\sin N\theta + b\cos N\theta + rA(r,\theta)),$$

for some a, b (not both zero) and some $N \geq 2$. Here we have used that u is non-constant and satisfies $\nabla\cdot(\gamma^{-1}\nabla u) = 0$ near x^*, and we have used that γ^{-1} is analytic (the case of a constant u is trivial). We may without loss of generality assume that the tangent to ρ at x^* is $\{(r,0),\ r > 0\} \cup \{(r,\pi),\ r > 0\}$. It follows that $b = 0$, *i.e.*,

$$u(x) = u(x^*) + ar^N(\sin N\theta + rA(r,\theta)).$$

Since u is analytic near x^*, it is well known that u as a function of (r, θ) is analytic on $[0, \epsilon] \times [0, 2\pi]$, the main point being that it is analytic at $r = 0$ and therefore also has an analytic extension to $[-\epsilon, \epsilon]$ for a sufficiently small ϵ (indeed the analytic extension for negative r is given by $\tilde{u}(r, \theta) = u(-r, \theta + \pi)$, $\theta + \pi$ taken modulo 2π). It follows that $A(r, \theta)$ also has an analytic extension near $r = 0$; we denote this extension by $\tilde{A}(r, \theta)$, $r \in [-\epsilon, \epsilon] \times [0, 2\pi]$. The function $F(r, \theta) = \sin N\theta + r\tilde{A}(r, \theta)$ satisfies

$$F(0, \pi/N) = 0 \quad \text{and} \quad \frac{\partial}{\partial \theta} F(0, \pi/N) = -N,$$

and therefore, by the implicit function theorem, it is possible to find a unique analytic function $\theta(r)$ such that $\theta(0) = \pi/N$ and $\{(r, \theta) : F(r, \theta) = 0\}$ coincides with $\{(r, \theta(r))\}$ in a neighborhood of $(0, \pi/N)$. The curve, ρ', given by

$$(r \cos \theta(r), r \sin \theta(r)) + x^*$$

is an analytic curve through x^*, w! ich satisfies $\rho' \cap \rho = x^*$ and which by its very definition is a level curve for u. □

3 Proof of Theorem 1.1

Let O be the open set enclosed by Σ and $\tilde{\Sigma}$, *i.e.*, the set of points in $\Omega \setminus (\Sigma \cup \tilde{\Sigma})$ from which it is only possible to reach $\partial\Omega$ by crossing Σ or $\tilde{\Sigma}$. Since $\Omega \setminus (O \cup \Sigma \cup \tilde{\Sigma})$ has only one connected component, it follows from the assumptions about the boundary data (by unique continuation) that

$$u_j = \tilde{u}_j \quad \text{in } \Omega \setminus (O \cup \Sigma \cup \tilde{\Sigma}) \quad j = 1, \ldots, M. \tag{3.1}$$

If O is nonempty then ∂O consists of pieces of curves from Σ and $\tilde{\Sigma}$; on each of these pieces either u_j or $\tilde{u}_j$, is constant. Due to (3.1) it now follows that u_j is constant on each of the pieces that make up ∂O. Each function u_j therefore assumes finitely many values on ∂O (at most $m+n$). Since u_j is continuous in Ω (cf. [1]) we get that u_j is constant on each connected component of ∂O. Each connected component of O is simply connected, and it now follows, by the maximum principle, that u_j is constant in each connected component of O. This implies that u_j is constant in all of Ω – a contradiction. We thus conclude that O is empty, so that $u_j = \tilde{u}_j$ in $\Omega \setminus (\Sigma \cup \tilde{\Sigma})$; by continuity it follows that

$$u_j = \tilde{u}_j \ \text{ in } \Omega \quad j = 1, \ldots, M. \tag{3.2}$$

Let us assume that Σ and $\tilde{\Sigma}$ are not identical. We may assume that there exists a curve ρ contained in $\tilde{\sigma}_k$ for some k with $\rho \cap \Sigma = \emptyset$. Based on (3.2) we conclude that the functions u_j are all constant on ρ. There must exist a point on ρ where $\nabla u_1 \neq 0$, since otherwise u_1 is constant in Ω (by unique continuation); the implicit function theorem asserts that ρ must be analytic near this point. By shortening, if necessary, we may assume that the entire curve ρ is analytic. Let ν be a unit normal vector field on the curve ρ and let $x_1, \ldots, x_{M-1}$ be distinct interior points on ρ. Let $\alpha_1, \ldots, \alpha_M$ denote numbers, *not all zero* , satisfying the underdetermined set of linear equations

$$\sum_{j=1}^{M} \frac{\partial u_j}{\partial \nu}(x_i)\alpha_j = 0, \qquad i = 1, \ldots, M-1.$$

Define the function

$$u(x) = \sum_{j=1}^{M} \alpha_j u_j(x) \tag{3.3}$$

for $x \in \Omega$. The curve ρ is also a level curve for u. Applying Lemma 2.1 we obtain an analytic curve ρ_0 containing ρ which satisfies (2.1a)–(2.1c), *i.e.*,

(3.4a) u is constant on ρ_0

(3.4b) ρ_0 has one endpoint on $\partial\Omega$ or σ_k for some k

(3.4c) ρ_0 has the other endpoint on $\partial\Omega$ or σ_l for some l and $l \neq k$.

For $x \in \rho_0$ we have $|\nabla u(x)| = |\partial u/\partial\nu(x)|$. ¿From equation (3.3) it follows that $\partial u/\partial\nu(x_i) = 0$, so that $\nabla u(x_i) = 0$ for $i = 1, \ldots, M-1$. Using Lemma 2.2 we may now for each of the critical points x_i construct an analytic curve ρ_i such that

(3.5a) $\rho_i \cap \rho_0 = \{x_i\}$,

(3.5b) u is constant on ρ_i.

Lemma 2.1 permits us to extend each of the curves ρ_i until it hits the boundary or one of the cracks in Σ, this way we obtain curves ρ_i which in addition to (3.5a) and (3.5b) satisfy

(3.5c) ρ_i has one endpoint on $\partial\Omega$ or σ_k for some k,

(3.5d) ρ_i has the other endpoint on $\partial\Omega$ or σ_l for some l and $l \neq k$.

The fact that the extended curve ρ_i still only intersects ρ_0 at x_i is proven as follows: if ρ_i intersected ρ_0 at some other point x_i' then there would be some nonempty region O enclosed by ρ_0 and ρ_i with u constant on ∂O. By the max-

imum principle u would be constant on O and hence it would be constant on all of Ω. This clearly contradicts the fact that $\gamma^{-1}\partial u/\partial\nu = \sum_{j=1}^{M} \alpha_j(\delta_{P_0} - \delta_{P_j})$ on $\partial\Omega$ where the P_j are distinct and at least one α_j is nonzero. Since all the curves ρ_i intersect ρ_0, the function u assumes the same constant value on $\cup_{i=0}^{M-1}\rho_i$. Note that no two of the $M-1$ curves $\rho_1,\ldots,\rho_{M-1}$ can intersect, for then we would have some nonempty region O enclosed by the M curves $\rho_0,\rho_1,\ldots,\rho_{M-1}$, with u constant on ∂O - a contradiction. For a similar reason none of the curves $\rho_0,\rho_1,\ldots,\rho_{M-1}$ can self-intersect. Between the M curves $\rho_0,\rho_1,\ldots,\rho_{M-1}$ we have a total of $2M$ endpoints. Note that no two of these curves can terminate on the same crack σ_k, for then we would have some region O bounded by these curves and the crack σ_k, with u constant on ∂O - a contradiction. There are n cracks in Σ, so it follows that there must be at least $2M-n$ points on $\partial\Omega$ at which the curves $\rho_0,\rho_1,\ldots,\rho_{M-1}$ terminate. Since the curves $\rho_1,\ldots,\rho_{M-1}$ do not intersect and each only intersect ρ_0 at one point it is easy to see that any connected component of $\Omega\setminus\cup_{i=0}^{M-1}\rho_i$ has a part of its boundary in common with $\partial\Omega$, and that the number of connected components is exactly equal to the number of terminal points of the curves $\rho_0,\rho_1,\ldots,\rho_{M-1}$ that lie on $\partial\Omega$ (at least $2M-n$). A situation corresponding to $n=2$ and $M=4$ is schematically shown in figure 1. The Neumann data for u has the form

$$\gamma^{-1}\frac{\partial u}{\partial\nu} = \sum_{j=0}^{M}\beta_j\delta_{P_j} \quad \text{on } \partial\Omega.$$

Let $M'+1 \le M+1$ be the number of nonzero $\beta's$ in the above sum, *i.e.*, $M'+1$ is the total number of sources and sinks (for u) on $\partial\Omega$. We note that none of the curves $\rho_0,\rho_1,\ldots,\rho_{M-1}$ can terminate at a source or a sink, since

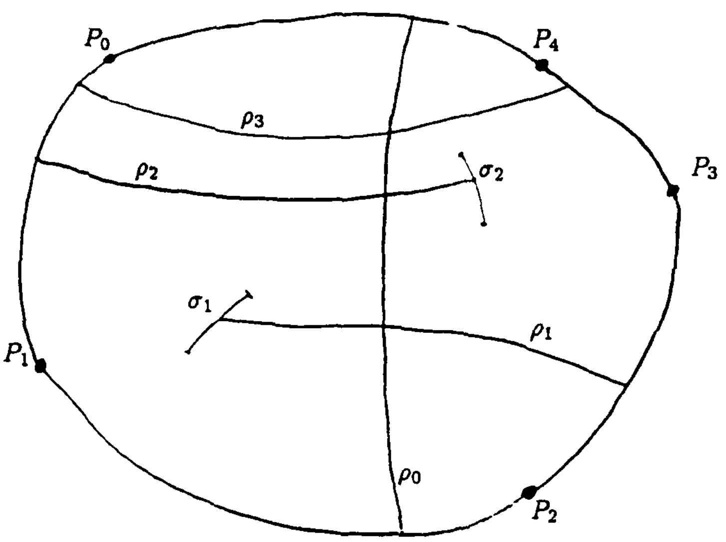

Figure 1:

$|u|$ approaches ∞ there.

If $M + (M - M') > n + 1$ then it follows that $2M - n = M + (M - M') - n + M' > M' + 1$, and therefore we conclude that at least one of the connected components of $\Omega \setminus \cup_{i=0}^{M-1} \rho_i$ is bounded by a level curve for u and a portion of $\partial\Omega$ on which the normal derivative of u vanishes. This forces u to be a constant – a contradiction. Hence we see that if $M + (M - M') > n + 1$, then the assumption that Σ and $\tilde{\Sigma}$ are different is incorrect.

Since $M \geq n+1$ and $M \geq M'$ we always have that $M + (M - M') \geq n+1$. The only case we have not analyzed yet is therefore $M + (M - M') = n + 1$, or equivalently, $M' = M = n + 1$. None of the curves $\rho_0, \rho_1, \ldots, \rho_{M-1}$ can now terminate at any of the points P_j, $j = 0, \ldots, M$ (since $|u|$ approaches ∞ there). Furthermore, each of the connected components of $\Omega \setminus \cup_{i=0}^{M-1} \rho_i$ has at least one of the points P_j on its boundary. If not, the argument from

the case $M + (M - M') > n + 1$ leads to a contradiction. There cannot be one connected component, the boundary of which contains two or more of the points P_j, because then, due to the identity $2M - n = M + 1$, there would automatically have to be one connected component, the boundary of which contained none of the points P_j – a contradiction. In summary, each connected component of $\Omega \setminus \cup_{i=0}^{M-1} \rho_i$ has exactly one of the points P_j on its boundary. This means that there are exactly $M + 1$ connected components and therefore exactly $M + 1$ terminal points of the curves $\rho_0, \rho_1, \ldots, \rho_{M-1}$ on $\partial\Omega$. This leaves $2M - (M+1) = M - 1 = n$ terminal points that fall on cracks – one on each crack of Σ. ¿From the inequality $n + 1 = M \geq \max(m, n) + 1$ we conclude that $n \geq m$. If $n = 0$ it would follow that $m = 0$, so that both $\Sigma = \tilde{\Sigma} = \emptyset$ – a contradiction. There is therefore at least one crack σ_{k_0} in the collection Σ. The crack σ_{k_0} is contained in the closure of one of the connected components of $\Omega \setminus \cup_{i=0}^{M-1} \rho_i$; we denote this connected component by O. Furthermore we denote by ρ' that part of $\cup_{i=1}^{M-1} \rho_i$ which connects σ_{k_0} to ρ_0. ρ' must necessarily, due to the construction of the curves ρ_i, be an interior boundary of O. A situation corresponding to n=2 and M=3 is illustrated in figure 2. Let P_{j_0} denote the point which lies on ∂O. We may without loss of generality assume that β_{j_0} is negative (so that P_{j_0} is a sink). Consider the maximum of u on $\overline{O}$. This maximum must be achieved at a boundary point, and it cannot be near P_{j_0}, since u behaves like $-\beta_{j_0}\gamma(P_{j_0})/\pi \ \log r$ there. Since $\frac{\partial u}{\partial \nu}$ is zero on $\partial\Omega \setminus \cup_{j=0}^{M} \{P_j\}$ it now follows from the strong version of the maximum principle that the maximum of u on $\overline{O}$ is attained on that part of ∂O which comes from $\cup_{i=0}^{M-1} \rho_i$; in particular the maximum is attained all along ρ'. Let x_0 be an interior point on ρ' and let $B \subset O \cup \rho'$ be

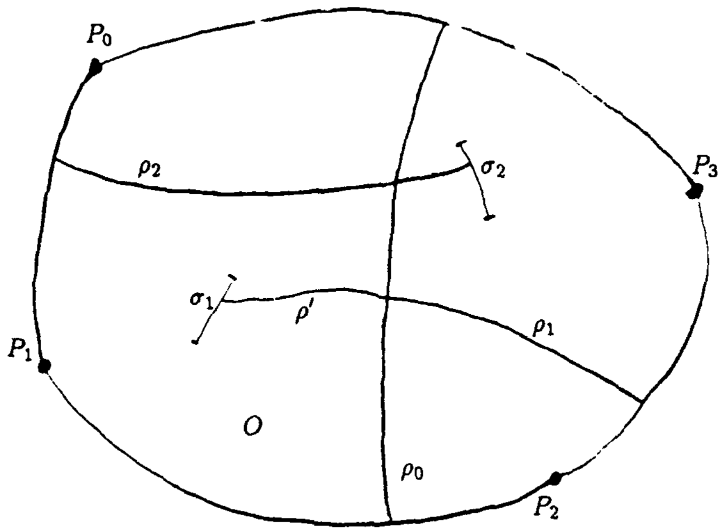

Figure 2:

a ball centered at x_0 which does not intersect Σ. The function u satisfies the elliptic equation $\nabla \cdot (\gamma^{-1}\nabla u) = 0$ in B (and is *not constant*) and therefore by the maximum principle

$$\inf_B u(x) < u(x_0) < \sup_B u(x). \tag{3.4}$$

On the other hand $B \subset \overline{O}$ so

$$u(x_0) = \max_{\overline{O}} u(x) \geq \sup_B u(x),$$

and this immediately leads to a contradiction with (3.4). Hence we conclude that also in the case $M + (M - M') = n + 1$ we cannot have that Σ and $\tilde{\Sigma}$ are different, and this completes the proof of Theorem 1.1. $\square$

4 Proof of Theorem 1.2

The proof of Theorem 1.2 goes entirely along the lines of the previous proof up to and including the construction of the function u and the curves ρ_i (using the equivalent of Lemma 2.1 and Lemma 2.2 with Dirichlet boundary conditions of the form $\sum \beta_j 1_{P_{j-1},P_j})$. From there the proof proceeds as outlined below.

The points $P_0, \ldots, P_M$ divide the boundary $\partial\Omega$ into $M+1$ half-open curves

$$[P_0, P_1), \ldots, [P_{M-1}, p_M), \text{ and } [P_M, P_0).$$

Here we have used the notation $[P, Q)$ for the counter-clockwise curve from P to Q, including P. The function u is constant on each of the curves $[P_0, P_1), \ldots, [P_{M-1}P_M)$, and $[P_M P_0)$ (on the last curve, u is actually zero). The curves $\rho_0, \ldots, \rho_{M-1}$ have at least $2M-n$ terminal points on the boundary of Ω, and we note that in this case the curves may very well terminate at one or more of the points $P_0, \ldots, P_M$. If $2M - n > M + 1$ (*i.e.*, $M > n+1$) it therefore follows that at least one of the curves $[P_0, P_1), \ldots, [P_{M-1}, P_M)$, and $[P_M, P_0)$ contains two terminal points of $\cup_{i=0}^{M-1} \rho_i$. Consequently there is a connected component of $\Omega \setminus \cup_{i=0}^{M-1} \rho_i$ which as its boundary has a level curve of u – a contradiction.

We now consider the remaining case: $M = n+1$. In this case we conclude that any one of the curves $[P_0, P_1), \ldots, [P_{M-1}, P_M)$, and $[P_M, P_0)$ contains exactly one terminal point of $\cup_{i=0}^{M-1} \rho_i$. This leaves $2M - (M+1) = M - 1 = n$ terminal points of the curves $\rho_0, \ldots, \rho_{M-1}$ that fall on cracks – one on each crack of Σ. We also see that there are exactly $M+1$ connected components

of $\Omega \setminus \cup_{i=0}^{M-1}\rho_i$. For each connected component, that part of the boundary which is shared with $\partial\Omega$ consists of a single curve between two adjacent terminal points of $\cup_{i=0}^{M-1}\rho_i$ (these points lie on adjacent curves $[P_{j-1}, P_j)$ and $[P_j, P_{j+1})$, indices counted modulo M+1). As in the proof of Theorem 1.1 we may argue that $n \geq 1$, so that Σ contains at least one crack σ_{k_0}. Let O denote the connected component of $\Omega \setminus \cup_{i=0}^{M-1}\rho_i$, whose closure contains σ_{k_0}, and let ρ' denote that part of $\cup_{i=0}^{M-1}\rho_i$, which connects σ_{k_0} to ρ_0. ρ' must necessarily, due to the construction of the curves ρ_i, be an interior boundary of O. The (interior) part of the boundary of O which is shared with $\partial\Omega$ consists of a curve from P to Q with $P \in [P_{j_0-1}, P_{j_0})$ and $Q \in [P_{j_0}, P_{j_0+1})$ for some j_0 (mod $M+1$). The rest of the boundary of O is a level curve for u. It is now very easy to see that u takes at most two values on ∂O. Consequently either the minimum or the maximum of u on $\overline{O}$ is attained on ρ'. This leads to a contradiction, just as in the proof of Theorem 1.1. $\square$

ACKNOWLEDGEMENT

This work was completed while the second author visited the Mathematical Sciences Research Institute, Berkeley, CA 94720. The MSRI is supported by a grant from the National Science Foundation.

References

[1] Friedman, A. and M. Vogelius, *Determining cracks by boundary measurements*, Indiana Univ. Math. J. 38 (1989), 527–556.

[2] Santosa, F. and M. Vogelius, *A computational algorithm to determine cracks from electrostatic boundary measurements*, to appear, Int. J. Eng. Sci.

Report Documentation Page

National Aeronautics and Space Administration

1. Report No. NASA CR-187583 ICASE Report No. 91-46	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A UNIQUENESS RESULT CONCERNING THE IDENTIFICATION OF A COLLECTION OF CRACKS FROM FINITELY MANY ELECTRO-STATIC BOUNDARY MEASUREMENTS		5. Report Date June 1991
		6. Performing Organization Code
7. Author(s) Kurt Bryan Michael Vogelius		8. Performing Organization Report No. 91-46
		10. Work Unit No. 505-90-52-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		11. Contract or Grant No. NAS1-18605
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Michael F. Card

Submitted to SIAM Journal of Mathematical Analysis

Final Report

16. Abstract

~~We~~ consider the problem of locating and identifying a collection of finitely many cracks inside a planar domain from measurements of the electrostatic boundary potentials induced by specified current fluxes. It is shown that a collection of n or fewer cracks can be uniquely identified by measuring the boundary potentials induced by n+1 specified current fluxes, consisting entirely of electrode pairs.

17. Key Words (Suggested by Author(s)) inverse problems, impedence imaging	18. Distribution Statement 59 - Mathematical and Computer Sciences (General) Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	18	A03

NASA FORM 1626 OCT 86

NASA-Langley, 1991